**Two species population model:** Let there are two interacting and competing species living in the same environment. If  and  are populations of these species, then the following system



where  and  are linear or nonlinear in the variables  and , is called a two species population model. Two species population models are two types such as

(a) Linear population model of two species and

(b) Non-linear population model of two species.

Example: (1). ,  is a linear population model of two species.

(2). ,  is a non-linear population model of two species.

**Non-linear population model of two species:** There are three well known non-linear population models of two species such as

1. **Lotka-Volterra Predator-Prey model:** If the growth rate of one population is decreased

and the other is increased, then the populations are in predator-prey situation (e.g., the populations of rats and cats or of rabbits and foxes etc.). Let  and  denote the populations of prey and predator respectively. Then the Lotka-Volterra predator-prey model is defined as



where , ,  and are positive constants.

In the absence of predators, the prey species increases and in the absence of preys, he predator species decreases. The contacts between predators and preys are harmful to the preys and are useful to the predators.

1. **Lotka-Volterra Competitive model:** If the growth rate of each population is decreased

then the populations are in competitive situation. Here two species compete for the same limited food source or in some way inhibit each other’s growth.For example,competition may be for territory which is directly related to food resources. When two species compete for the same limited resources one of the species usually becomes extinct.

Let  and  denote the competing species at any time . Then the Lotka-Volterra competitive model is defined as



where , ,  and are positive constants.

1. **Lotka-Volterra mutualism or symbiosis model:** If the growth rate of each population is

enhancedthen the populations are in mutualism or symbiosis situation. There are many examples where the interaction of two or more species is to the advantage of all. For example, flowering plants being pollinated by animals, vascular plants being dispersed by animals.

Let  and  denote the symbiotic species at any time . Then the Lotka-Volterra mutualism or symbiosis model is defined as



where , ,  and are positive constants.

**Question-01:** Discuss two species Lotka-Volterra Predator-Prey model and find its exact solution. Make a stability analysis of this model and sketch its phase portrait.

**Answer:** The system governed by two nonlinear ordinary differential equations



where , ,  and are positive constants, is called the two species Lotka-Volterra Predator-Prey model. Here is the prey population and  is the predator population at any time ,  and  are the specific growth rates of  and respectively,  and  are the coefficients of predation. The first equation of (1) is called the prey equation and the second equation of (1) is called the predator equation.

**Explanation:** From the prey equation we observe that in the absence of predator, the prey

population increases and in the presence of predator, the prey population decreases. Similarly, from the predator equation we observe that in the absence of prey, the predator population decreases and in the presence of prey, the predator population increases.

**Stability analysis:** The system (1) cannot be solved explicitly and so the exact paths of (1)

cannot be drawn in the plane. However we can find equilibrium points where the population remains constant.

For equilibrium or critical points putting  and  in (1), we have



Solving (2) we find the equilibrium points  and . We now investigate the stability of these equilibrium points.

**Case-I: Equilibrium point :** This is the case where both the species are absent.

Thus when the populations are sufficiently close to this points, we can neglect the second terms  and  of (1) in comparison to the first terms. Hence (1) reduces to



The solutions of (3) are



where  and  are arbitrary constants. Since  and  are non-negative so we must have  and  and the family of curves described by (4) is shown in figure -1.

Figure-1

Further, if we displace the population slightly from the equilibrium point ****, then it tends to move away from the point **.** Thus the equilibrium point ****is unstable.

**Case-II: Equilibrium point :** This is the case where both the species are

present. We linearize (1) by using



This transforms the point **** to the origin ****and makes **** and  small. Putting (5) in (1) we obtain,



Since  and  are small so we can neglect the terms of  and then we obtain



The characteristic equation of (7) is,







Hence **** is a stable centre.

Again from (7), we have





Integrating, 



where  is an integrating constant. This represents a family of concentric ellipses. We find from (7) that the directions of these ellipses are anti-clockwise. Hence the stable centre **** is shown in figure-2.

Figure-2

**Exact/ implicit solution:** From (1), we have







Integrating, 









where  is an integrating constant.

This is the general solution of (1).

Using initial populations and , we get



Putting this value in (8), we get



This is the particular solution of (1).

Now let  and . Then from (8), we get



Looking carefully at , we note that

1.  when 
2.  when 

Also 

and 

Thus the critical value of  is  and  is  at 

Hence  has a local maximum at . Thus  looks like the following figure-3

Figure-3

Obviously will have a similar curve. Since the product of  and is a constant so  looks like of the following figure-4

Figure-4

Figure-5

This implies that the path of (1) is a closed loop as shown in figure-5. Hence from figures 1, 2 and 5 the population dynamics of (1) near its equilibrium points are shown in the following phase plane.

Figure-6

**Question-02:** Discuss two species Lotka-Volterra competitive model and find its exact solution. Make a stability analysis of this model and sketch its phase portrait.

**Answer:** The system governed by two nonlinear ordinary differential equations



where , ,  and are positive constants, is called the two species Lotka-Volterra competitive model with no carrying capacity. Here  and  are the two competing species at any time ,  and  are the specific growth rates of  and respectively,  and  are the coefficients of competition.

**Explanation:** From the 1st equation of (1) we observe that in the absence of , the species

 increases and in the presence of , the species  decreases. Similarly, from the 2nd equation of (1) we observe that in the absence of , the species  increases and in the presence of , the species  decreases. Thus the presence of each has an inhibiting effect upon the growth of the other species.

**Stability Analysis:** The system (1) cannot be solved explicitly and so the exact paths of

(1) cannot be drawn in the plane. However we can find equilibrium points where the population remains constant.

For equilibrium or critical points putting  and  in (1), we have



Solving (2) we find the equilibrium points  and . We now investigate the stability of these equilibrium points.

**Case-I: Equilibrium point :** This is the case where both the species are absent.

Thus when the populations are sufficiently close to this points, we can neglect the second terms  and  of (1) in comparison to the first terms. Hence (1) reduces to



The solutions of (3) are



where  and  are arbitrary constants. Since  and  are non-negative so we must have  and  and the family of curves described by (4) is shown in figure -1.

Figure-1

Further, if we displace the population slightly from the equilibrium point ****, then it tends to move away from **.** Thus the equilibrium point ****is unstable.

**Case-II: Equilibrium point :** This is the case where both the species are

present. We linearize (1) by using



This transforms the point **** to the origin ****and makes **** and  small. Putting (5) in (1) we obtain,



Since  and  are small so we can neglect the terms of  and then we obtain



The characteristic equation of (7) is,







The roots are real, unequal and of opposite signs.

Hence **** is an unstable saddle.

Again from (7), we have





Integrating, 



where  is an integrating constant. This represents a family of hyperbolas. The unstable saddle **** is shown in figure-2.

Figure-2

**Exact/ implicit solution:** From (1), we have







Integrating, 

where  is an integrating constant.

This is the general solution of (1).

Using initial populations and , we get



Putting this value in (8), we get





This is the particular solution of (1).

To find the equation of , we note that the left side of (9) has a maximum at  and the right side of (9) has a maximum at . The separatrix occurs when the initial values have the critical value ****. Thus is given by





The equation (10) represents the curve  for **** or  for **.** the curves of (9) and (10) are shown in figure-3.

Figure-3

**Question-03:** Discuss two species Lotka-Volterra mutualism or symbiosis model and find its exact solution. Make a stability analysis of this model and sketch its phase portrait.

**Answer:** The system governed by two nonlinear ordinary differential equations



where , ,  and are positive constants, is called the two species Lotka-Volterra Symbiotic model. Here and  are symbiotic populations at any time ,  and  are the specific growth rates of  and respectively,  and  are the coefficients of cooperation.

**Explanation:** From the first equation of (1) we observe that in the absence of , the

species  decreases exponentially and in the presence of , the species  increases. Similarly, from the second equation of (1) we observe that in the absence of , the species  decreases exponentially and in the presence of , the species  increases. Thus the presence of each species has a cooperative effect upon the growth of the other species.

**Stability analysis:** The system (1) cannot be solved explicitly and so the exact paths of (1)

cannot be drawn in the plane. However we can find equilibrium points where the population remains constant.

For equilibrium or critical points putting  and  in (1), we have



Solving (2) we find the equilibrium points  and . We now investigate the stability of these equilibrium points.

**Case-I: Equilibrium point :** This is the case where both the species are absent.

Thus when the populations are sufficiently close to this points, we can neglect the second terms  and  of (1) in comparison to the first terms. Hence (1) reduces to



The solutions of (3) are



where  and  are arbitrary constants. Since  and  are non-negative so we must have  and  and the family of curves described by (4) is shown in figure -1.

Figure-1

Further, if we displace the population slightly from the equilibrium point ****, then it tends to move towards and enters the equilibrium point **.** Thus the equilibrium point ****is asymptotically stable.

**Case-II: Equilibrium point :** This is the case where both the species are

present. We linearize (1) by using



This transforms the point **** to the origin ****and makes **** and  small. Putting (5) in (1) we obtain,



Since  and  are small so we can neglect the terms of  and then we obtain



The characteristic equation of (7) is,







The roots are real, unequal and of opposite signs.

Hence **** is an unstable saddle.

Again from (7), we have





Integrating, 



where  is an integrating constant. This represents a family of hyperbolas. Hence from (8) and figure-1, the saddle **** is shown in figure-2.

Figure-2

**Exact/ implicit solution:** From (1), we have







Integrating, 

where  is an integrating constant.

This is the general solution of (1).

Using initial populations and , we get



Putting this value in (9), we get









This is the particular solution of (1).

To find the equation of , we note that the left side of (10) has a maximum at  and the right side of (10) has a maximum at . The separatrix occurs when the initial values have the critical value ****. Thus is given by





The equation (11) represents the curve  for **** or  for **.** the curves of (10) and (11) are shown in figure-3.

Figure-3

**Problem-01:** Examine the nature and stability of the critical points of the system

****

and sketch its phase portrait.

**Solution:** The given system is

****

**Identification:** From the 1st equation of (1), we observe that in the absence of **, ** increases logistically and in the presence of **,** the growth rate of **** is diminished. Similarly, from the 2nd equation of (1) we observe that in the absence of **, **increases logistically and in the presence of **,** the growth rate of ****is diminished.

Hence this is a two species Lotka-Volterra competition model.

**2nd Part:** For equilibrium or critical points putting  and  in (1), we get

****

From (2),  and 

From (3),  and 

Solving (4) and (5), we get

, 

From (5), putting , we get 

From (4), putting , we get 

Therefore the critical points or equilibrium points are , , and .

**Case-01:** For the critical point , putting the transformation  ,  in (1) , we get

****

The corresponding linear system is,

****

The characteristic equation of (6) is,

****

****

****

Since the roots are real, unequal and both positive, so the critical point is an unstable node.

**Case-02:** For the critical point , putting the transformation  , in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (7) is,

****

****

****

****

Since the roots are real, unequal and of opposite sign, so the critical point is an unstable saddle.

**Case-03:** For the critical point , putting the transformation  , in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (8) is,

****

****

****

Since the roots are real, unequal and both negative, so the critical point is an asymptotically stable node.

**Case-04:** For the critical point , putting the transformation  , in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (9) is,

****

****

****

Since the roots are real, unequal and both negative, so the critical point is an asymptotically stable node.

**Phase portrait:** we have from (1)



The isoclines of (10) are given by



where  represents the slopes of the integral curves of (10).

when ,  then  and 

when ,  then  and 

The phase portrait is as follows:

Figure

**Problem-02:** Discuss the ecological model

****

and sketch its phase portrait.

**Solution:** The given system is

****

**Identification:** From the 1st equation of (1), we observe that in the absence of **, ** increases logistically and in the presence of **,** the growth rate of **** is diminished. Similarly, from the 2nd equation of (1) we observe that in the absence of **, **increases logistically and in the presence of **,** the growth rate of ****is diminished.

Hence this is a two species Lotka-Volterra competition model.

**2nd Part:** For equilibrium or critical points putting  and  in (1), we get

****

From (2),  and 

From (3),  and 

Solving (4) and (5), we get

, 

From (5), putting , we get 

From (4), putting , we get 

Therefore the critical points or equilibrium points are , , and .

**Case-01:** For the critical point , putting the transformation  ,  in (1) , we get

****

The corresponding linear system is,

****

The characteristic equation of (6) is,

****

****

****

Since the roots are real, unequal and both positive, so the critical point is an unstable node.

**Case-02:** For the critical point , putting the transformation  , in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (7) is,

****

****

****

****

Since the roots are real, unequal and of opposite sign, so the critical point is an unstable saddle.

**Case-03:** For the critical point , putting the transformation  , in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (8) is,

****

****

****

Since the roots are real, unequal and both negative, so the critical point is an asymptotically stable node.

**Case-04:** For the critical point , putting the transformation  ,  in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (9) is,

****

****

****

Since the roots are real, unequal and both negative, so the critical point is an asymptotically stable node.

**Phase portrait:** we have from (1)



The isoclines of (10) are given by



where  represents the slopes of the integral curves of (10).

when ,  then  and 

when ,  then  and 

The phase portrait is as follows:

Figure

**Problem-03:** Identify and draw the phase portrait of the system

**.**

**Solution:** The given system is

****

**Identification:** From the 1st equation of (1), we observe that in the absence of **, ** increases logistically and in the presence of **,** the growth rate of **** is diminished. Similarly, from the 2nd equation of (1) we observe that in the absence of **, **decreases exponentially and in the presence of **,** the growth rate of ****increases.

Hence this is a two species Lotka-Volterra Prey-Predator model.

**2nd Part:** For equilibrium or critical points putting  and  in (1), we get

****

From (2),  and 

From (3),  and 

Solving (4) and (5), we get

, 

From (4), putting , we get 

Therefore the critical points or equilibrium points are ,  and .

**Case-01:** For the critical point , putting the transformation  ,  in (1) , we get

****

The corresponding linear system is,

****

The characteristic equation of (6) is,

****

****

****

Since the roots are real, unequal and of opposite sign, so the critical point is an unstable saddle.

**Case-02:** For the critical point , putting the transformation  ,  in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (7) is,

****

****

****

****

Since the roots are complex with negative real part, so the critical point is an asymptotically stable spiral.

**Case-03:** For the critical point , putting the transformation  ,  in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (8) is,

****

****

****

Since the roots are real, unequal and of opposite sign, so the critical point is an unstable saddle.

**Phase portrait:** we have from (1)



The isoclines of (9) are given by



where  represents the slopes of the integral curves of (10).

when ,  then  and 

when ,  then  and 

The phase portrait is as follows:

Figure

**Problem-04:** Identify and draw the phase portrait of the system

**.**

**Solution:** The given system is

****

**Identification:** From the 1st equation of (1), we observe that in the absence of **, ** increases logistically and in the presence of **,** the growth rate of **** is diminished. Similarly, from the 2nd equation of (1) we observe that in the absence of **, **increases logistically and in the presence of **,** the growth rate of ****is diminished.

Hence this is a two species Lotka-Volterra competition model.

**2nd Part:** For equilibrium or critical points putting  and  in (1), we get

****

From (2),  and 

From (3),  and 

Solving (4) and (5), we get

, 

From (5), putting , we get 

From (4), putting , we get 

Therefore the critical points or equilibrium points are , , and .

**Case-01:** For the critical point , putting the transformation  ,  in (1) , we get

****

The corresponding linear system is,

****

The characteristic equation of (6) is,

****

****

****

Since the roots are real, unequal and both positive, so the critical point is an unstable node.

**Case-02:** For the critical point , putting the transformation  , in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (7) is,

****

****

****

****

Since the roots are real, unequal and of opposite sign, so the critical point is an unstable saddle.

**Case-03:** For the critical point , putting the transformation  ,  in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (8) is,

****

****

****

Since the roots are real, unequal and both negative, so the critical point is an asymptotically stable node.

**Case-04:** For the critical point , putting the transformation  ,  in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (9) is,

****

****

****

Since the roots are real, unequal and both negative, so the critical point is an asymptotically stable node.

**Phase portrait:** we have from (1)



The isoclines of (10) are given by



where  represents the slopes of the integral curves of (10).

when ,  then  and 

when ,  then  and 

The phase portrait is as follows:

Figure

**Problem-05:** Identify the population model

****

Find the equilibrium population and sketch its phase portrait.

**Solution:** The given system is

****

**Identification:** From the 1st equation of (1), we observe that in the absence of **, ** increases logistically and in the presence of **,** the growth rate of **** is diminished. Similarly, from the 2nd equation of (1) we observe that in the absence of **, **increases logistically and in the presence of **,** the growth rate of ****increases.

Hence this is a two species Lotka-Volterra prey-predator model.

**2nd Part:** For equilibrium or critical points putting  and  in (1), we get

****

From (2),  and 

From (3),  and 

Solving (4) and (5), we get

, 

From (5), putting , we get 

From (4), putting , we get 

Therefore the critical points or equilibrium points are , , and .

**Case-01:** For the critical point , putting the transformation  ,  in (1) , we get

****

The corresponding linear system is,

****

The characteristic equation of (6) is,

****

****

****

Since the roots are real, unequal and of opposite sign, so the critical point is an unstable saddle.

**Case-02:** For the critical point , putting the transformation  , in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (7) is,

****

****

****

Since the roots are purely imaginary, so the critical point is stable centre.

**Case-03:** For the critical point , putting the transformation  ,  in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (8) is,

****

****

****

Since the roots are real, unequal and of opposite sign, so the critical point is an unstable saddle.

**Case-04:** For the critical point , putting the transformation  ,  in (1), we get

****

****

The corresponding linear system is

****

The characteristic equation of (9) is,

****

****

****

Since the roots are real, unequal and of opposite sign, so the critical point is an unstable saddle.

**Phase portrait:** we have from (1)



The isoclines of (10) are given by



where  represents the slopes of the integral curves of (10).

when ,  then  and 

when ,  then  and 

The phase portrait is as follows:

Figure

**Problem-06:** Identify and solve the linear population model



where . When the species will be eliminated?

**Solution:** We have



where .

**Identification:** From the 1st equation of (1), we observe that in the absence of **, ** increases exponentially and in the presence of **,** the growth rate of **** is also enhanced. Similarly, from the 2nd equation of (1) we observe that in the absence of **, **increases exponentially and in the presence of **,** the growth rate of ****i**s** diminished.

Hence this is a two species predator-prey model.

**2nd part:** The given model can be written as



where  and 

The characteristic equation is















The eigen values are .

**Case-01:** Let  be the eigen vector corresponding to the value . Then we have,













A simple non trivial solution is, 



**Case-02:** Let  be the eigen vector corresponding to the value . Then we have,













A simple non trivial solution is, 



Hence the matrix  with the eigen vectors  and  is given by







Since the eigen values are unequal there exist a diagonal matrix .









Thus 





Hence the solution of (1) is,











and 

**3rd part:** The species  will be eliminated if .

i.e. 







**Problem-07:** Identify and solve the linear population model



where . Which population will be eliminated and find the elimination time? What will the population after 6 months?

**Solution:** We have



where .

**Identification:** From the 1st equation of (1), we observe that in the absence of **, ** increases exponentially and in the presence of **,** the growth rate of **** is diminished. Similarly, from the 2nd equation of (1) we observe that in the absence of **, **increases exponentially and in the presence of **,** the growth rate of ****i**s** diminished.

Hence this is a two species competition model.

**2nd part:** The given model can be written as



where  and 

The characteristic equation is









The eigen values are 

**Case-01:** Let  be the eigen vector corresponding to the value . Then we have,













A simple non trivial solution is, .



**Case-02:** Let  be the eigen vector corresponding to the value . Then we have,













A simple non trivial solution is, 



Hence the matrix  with the eigen vectors  and  is given by







Since the eigen values are unequal there exist a diagonal matrix .











Thus 







Hence the solution of (1) is,













and 

Which represents all future population. The population  will be eliminated.

The will be eliminated if .

i.e. 











When  months year, we get



and .

**Problem-08:** Identify and solve the linear population model



where . When the species  will be eliminated?

**Solution:** We have



where .

**Identification:** From the 1st equation of (1), we observe that in the absence of **, ** increases exponentially and in the presence of **,** the growth rate of **** is diminished. Similarly, from the 2nd equation of (1) we observe that in the absence of **, **increases exponentially and in the presence of **,** the growth rate of ****i**s** also enhanced.

Hence this is a two species predator-prey model.

**2nd part:** The given model can be written as



where  and 

The characteristic equation is









The eigen values are 

**Case-01:** Let  be the eigen vector corresponding to the value . Then we have,















A simple non trivial solution is, .



**Case-02:** Let  be the eigen vector corresponding to the value . Then we have,

















A simple non trivial solution is, 



Hence the matrix  with the eigen vectors  and  is given by







Since the eigen values are unequal there exist a diagonal matrix .















Thus 









Hence the solution of (1) is,













and 

Which is the required population at any.

The population will be eliminated if .

i.e. 





 years.

**Problem-09:** Identify and solve the linear population model



where . By the principal matrix method.

**Solution:** We have



where .

**Identification:** From the 1st equation of (1), we observe that in the absence of **, ** increases exponentially and in the presence of **,** the growth rate of **** is diminished. Similarly, from the 2nd equation of (1) we observe that in the absence of **, **increases exponentially and in the presence of **,** the growth rate of ****i**s** also enhanced.

Hence this is a two species predator-prey model.

**2nd part:** The given model can be written as



where  and 

The characteristic equation is









The eigen values are 

**Case-01:** Let  be the eigen vector corresponding to the value . Then we have,











A simple non trivial solution is, .



**Case-02:** Let  be the eigen vector corresponding to the value . Then we have,











A simple non trivial solution is, 



Hence the matrix  with the eigen vectors  and  is given by







The Jordan matrix  is given by,











Thus 









Hence the solution of (1) is,











and 

Which is the required population at any.

The population will be eliminated if .

i.e. 





.

**Problem-10:** Identify and solve the linear population model



where . When the species  will be eliminated.

**Solution:** We have



where .

**Identification:** From the 1st equation of (1), we observe that in the absence of **, ** increases exponentially and in the presence of **,** the growth rate of **** is diminished. Similarly, from the 2nd equation of (1) we observe that in the absence of **, **increases exponentially and in the presence of **,** the growth rate of ****i**s** also enhanced .

Hence this is a two species prey-predator model.

**2nd part:** The given model can be written as



where  and 

The characteristic equation is















The eigen values are .

**Case-01:** Let  be the eigen vector corresponding to the value . Then we have,













A simple non trivial solution is, 



**Case-02:** Let  be the eigen vector corresponding to the value . Then we have,













A simple non trivial solution is, 



Hence the matrix  with the eigen vectors  and  is given by







Since the eigen values are unequal so there exist a diagonal matrix .

















Thus 









Hence the solution of (1) is,













and 

Which is the required population at any.

The population will be eliminated if .

i.e. 





 years.